

REVERSIBILITY IN THE GROUPS $PL^+(\mathbb{S}^1)$ AND $PL(\mathbb{S}^1)$

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ABSTRACT. Let $PL^+(\mathbb{S}^1)$ be the group of order preserving piecewise linear homeomorphisms of the circle. An element in $PL^+(\mathbb{S}^1)$ is called *reversible* in $PL^+(\mathbb{S}^1)$ if it is conjugate to its inverse in $PL^+(\mathbb{S}^1)$. We characterize the reversible elements in $PL^+(\mathbb{S}^1)$. We also perform a similar characterisation in the full group $PL(\mathbb{S}^1)$ of piecewise linear homeomorphisms of the circle.

1. Introduction

Let G be a group. An element $g \in G$ is called *reversible* in G if it is conjugate to its inverse in G ; there exists $h \in G$ such that

$$(1.1) \quad g^{-1} = hgh^{-1}.$$

We say that g is reversed by h . An element h in G is called an *involution* if $h = h^{-1}$. If the conjugating element h can be chosen to be an involution then g is called *strongly reversible*. Any strongly reversible element can be expressed as a product of two involutions. So involutions are strongly reversible and strongly reversible elements are reversible. For $n \in \mathbb{N}^*$, define

$$I_n(G) = \{\tau_1\tau_2 \dots \tau_n \quad : \quad \forall 1 \leq i \leq n, \tau_i \text{ is an involution in } G\};$$

$$R_n(G) = \{g_1g_2 \dots g_n \quad : \quad \forall 1 \leq i \leq n, g_i \text{ is a reversible element in } G\}.$$

For each integer $n \in \mathbb{N}^*$, it is clear that $I_n(G) \subseteq I_{n+1}(G)$ and $R_n(G) \subseteq R_{n+1}(G)$. The set $I_1(G)$ (resp. $I_2(G)$) consists of the involutions (resp. the strongly reversible elements) in G . Denote by

- $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ the circle, it is a multiplicative group.
- $\text{Homeo}(\mathbb{S}^1)$ (resp. $\text{Homeo}^+(\mathbb{S}^1)$) the group of all homeomorphisms (resp. orientation-preserving homeomorphisms) of \mathbb{S}^1 .
- id the identity map of \mathbb{S}^1 .
- $s : \mathbb{S}^1 \longrightarrow \mathbb{S}^1, z \mapsto \bar{z}$ the reflection.
- $\text{Fix}(f)$ the set of fixed points of f .

Two elements f and g of $\text{Homeo}(\mathbb{S}^1)$ are called *conjugate* in $\text{Homeo}(\mathbb{S}^1)$ if there exists $h \in \text{Homeo}(\mathbb{S}^1)$ such that $g = hfh^{-1}$. It is well known that for

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every homeomorphism $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ there exists a unique (up to a translation by an integer) homeomorphism $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$(1.2) \quad f(e^{i2\pi x}) = e^{i2\pi \tilde{f}(x)} \text{ and } \tilde{f}(x+1) = \tilde{f}(x) + k, \text{ for all } x \in \mathbb{R}$$

where $k \in \{-1, 1\}$. Such a homeomorphism \tilde{f} is called a *lift* of f . We call f orientation-preserving if $k = 1$; resp. orientation-reversing if $k = -1$; which is equivalent to the fact that \tilde{f} is increasing, resp. \tilde{f} is decreasing.

Definition 1.1. A homeomorphism f of \mathbb{S}^1 is said to be *piecewise linear* (PL) if it is derivable except at finitely or countably many points $(c_i)_{i \in \mathbb{N}}$ called *break points* of f at which f admits left and right derivatives (denoted, respectively, by Df_- and Df_+) and such that the derivative $\text{Df} : \mathbb{S}^1 \rightarrow \mathbb{R}^*$ is constant on each connected component of $\mathbb{S}^1 \setminus \{c_i : i \in \mathbb{N}\}$.

Let f be a *piecewise linear* (PL) homeomorphism f of \mathbb{S}^1 . Define $\sigma_f(x) = \frac{\text{Df}_-(x)}{\text{Df}_+(x)}$ the f -jump in $x \in \mathbb{S}^1$. Denote by $B(f) = \{c_i : i \in \mathbb{N}\}$ the set of *break points* of f . So $B(f) = \{x \in \mathbb{S}^1 : \sigma_f(x) \neq 1\}$.

The homeomorphism f is PL (resp. PL^+ , PL^-) if and only if \tilde{f} is a piecewise linear (resp. piecewise linear increasing, piecewise linear decreasing) homeomorphism of the real line \mathbb{R} .

Denote by

- $\text{PL}(\mathbb{S}^1)$ the group of all piecewise linear homeomorphisms of \mathbb{S}^1 ,
- $\text{PL}^+(\mathbb{S}^1)$ the group of orientation-preserving elements of $\text{PL}(\mathbb{S}^1)$,
- $\text{PL}^-(\mathbb{S}^1)$ the set of orientation-reversing elements of $\text{PL}(\mathbb{S}^1)$.

If $f \in \text{Homeo}^+(\mathbb{S}^1)$, we denote by $\rho(f)$ its rotation number.

In the sequel we identify $\rho(f)$ to its lift in $[0, 1[$.

It is known (see for instance [5]) that if an element $f \in \text{PL}^+(\mathbb{S}^1)$ is reversed by $h \in \text{PL}^+(\mathbb{S}^1)$, then by equality (1.1), $\rho(f) = 0$ or $\frac{1}{2}$.

We refer the reader to the book [9] and papers ([4], [8]) for a thorough account on groups of circle homeomorphisms.

The object of this paper is to characterize reversible elements (resp. strongly reversible elements) in the groups $\text{PL}^+(\mathbb{S}^1)$ and $\text{PL}(\mathbb{S}^1)$. Our main results are the following.

Theorem 1.2 (Reversibility in $\text{PL}^+(\mathbb{S}^1)$). *Let $f \in \text{PL}^+(\mathbb{S}^1)$. Then f is reversible in $\text{PL}^+(\mathbb{S}^1)$ if and only if one of the following holds:*

- (1) $\rho(f) = 0$, and f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$.
- (2) $\rho(f) = \frac{1}{2}$, and f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.

Remark 1.3. If instead of the group $PL^+(\mathbb{S}^1)$ we take the group $\text{Homeo}^+(\mathbb{S}^1)$, the theorem 1.2 is false: In [1], Gill et al. gave an example of a homeomorphism $f \in \text{Homeo}^+(\mathbb{S}^1)$ with rotation number $\rho(f) = \frac{1}{2}$ that is reversible in $\text{Homeo}^+(\mathbb{S}^1)$ but not strongly reversible in $\text{Homeo}(\mathbb{S}^1)$.

Theorem 1.4 (Reversibility in $PL(\mathbb{S}^1)$). *In $PL(\mathbb{S}^1)$ reversibility and strong reversibility are equivalent.*

We denote by n_f the smallest positive integer n such that f^n has fixed points and by Δ_f the signature of f (see definition in Section 2).

Theorem 1.5. (1) *Let $f \in PL^+(\mathbb{S}^1)$. Then f is strongly reversible in $PL^+(\mathbb{S}^1)$ if and only if one of the following holds.*

- (i) $f^2 = \text{id}$.
- (ii) $\text{Fix}(f) \neq \emptyset$ and there exists $h \in PL^+(\mathbb{S}^1)$ such that $\rho(h) = \frac{1}{2}$ and $\Delta_f = -\Delta_f \circ h$.

(2) *Let $f \in PL^+(\mathbb{S}^1)$.*

(i) *If $\rho(f) \in \mathbb{Q}$ then f is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$ if and only if there exists an involution $h \in PL^-(\mathbb{S}^1)$ such that $\Delta_{f^{n_f}} = \Delta_{f^{n_f}} \circ h$.*

(ii) *If $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$ then f is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$ if and only if f is conjugate to the rotation $r_{\rho(f)}$ through a homeomorphism h such that $hrsh^{-1} \in PL^-(\mathbb{S}^1)$, for some rotation r of \mathbb{S}^1 .*

The next theorem is about composition of reversible (resp. involution) maps.

Theorem 1.6. *We have*

- (i) $PL^+(\mathbb{S}^1) = R_2(PL^+(\mathbb{S}^1)) = I_3(PL^+(\mathbb{S}^1)) \neq I_2(PL^+(\mathbb{S}^1))$ and $R_1(PL^+(\mathbb{S}^1)) \subsetneq I_2(PL(\mathbb{S}^1))$.
- (ii) $PL(\mathbb{S}^1) = R_2(PL(\mathbb{S}^1)) = I_3(PL(\mathbb{S}^1)) \neq I_2(PL(\mathbb{S}^1))$ and $R_1(PL(\mathbb{S}^1)) = I_2(PL(\mathbb{S}^1)) \neq I_1(PL(\mathbb{S}^1))$.

The structure of the paper is as follows. In Section 2 we give some notations and preliminaries results that are needed for the rest of the paper. In Section 3 we study reversibility in $PL^+(\mathbb{S}^1)$ of elements f of $PL^+(\mathbb{S}^1)$ by proving Theorem 1.2. In Section 4, we study reversibility in $PL(\mathbb{S}^1)$ by proving Theorem 1.4. Section 5 is devoted to the characterisation of strong reversibility in $PL(\mathbb{S}^1)$ of elements of $PL^+(\mathbb{S}^1)$. Finally, Section 6 is devoted to the proof of Theorem 1.6.

2. Notations and some results

Denote by

- T the translation of \mathbb{R} defined by $T(x) = x + 1$, and for each $a \in \mathbb{R}$ let T_a be the translation defined by $T_a(x) = x + a$. So $T = T_1$.

- $r_\pi : z \mapsto -z$ the rotation of \mathbb{S}^1 by π .

- For points x and y in \mathbb{S}^1 , we denote by (x, y) the open anticlockwise interval from x to y , and by $[x, y]$ the closure of (x, y) . We say that $x < y$ in a proper open interval I in \mathbb{S}^1 if $(x, y) \subset I$.

- For $f \in \text{Homeo}(\mathbb{S}^1)$ we denote by $\deg(f) = \begin{cases} 1, & \text{if } f \in \text{Homeo}^+(\mathbb{S}^1) \\ -1, & \text{if } f \in \text{Homeo}^-(\mathbb{S}^1) \end{cases}$

the degree of f .

- For $f \in \text{Homeo}^+(\mathbb{S}^1)$ which has a fixed point, then each point $x \in \mathbb{S}^1$ is either in $\text{Fix}(f)$ or it lies in an open interval component I in $\mathbb{S}^1 \setminus \text{Fix}(f)$. The *signature* of f (see [1]) is a map $\Delta_f : \mathbb{S}^1 \rightarrow \{-1, 0, 1\}$ given by

$$\Delta_f(x) = \begin{cases} 1, & \text{if } f(x) > x \\ 0, & \text{if } f(x) = x \\ -1, & \text{if } f(x) < x. \end{cases}$$

We have the following lemma.

Lemma 2.1. [1] *Let $f \in \text{Homeo}^+(\mathbb{S}^1)$ with a fixed point and $h \in \text{Homeo}(\mathbb{S}^1)$. Then*

(i) $\Delta_{hfh^{-1}} = \deg(h)(\Delta_f \circ h^{-1})$.

(ii) $\Delta_{f^{-1}} = -\Delta_f$.

Lemma 2.2. *Let $f, h \in \text{Homeo}(\mathbb{S}^1)$ be such that $f^{-1} = hfh^{-1}$ and let $n \in \mathbb{Z}$. Then we have $f^{(-1)^n} = h^n fh^{-n}$.*

Proof. The proof is down by induction, which is straightforward. □

The following lemma shows that any reversible element of $\text{PL}^+(\mathbb{R})$ must have a fixed point.

Lemma 2.3. *Let $f, h \in \text{PL}^+(\mathbb{R})$ such that $hfh^{-1} = f^{-1}$. Assume that f is not the identity. Then $\text{Fix}(f) \neq \emptyset$ and $\text{Fix}(h) = \emptyset$.*

Proof. In fact we show that f has a fixed point in any subinterval I of \mathbb{R} such that $f(I) = I = h(I)$: Otherwise, either $f(x) > x$ for all $x \in I$ or $f(x) < x$ for all $x \in I$. Assume that $f(x) > x$ for all $x \in I$. Then for all $x \in I$, $f^{-1}(x) = hf(h^{-1}(x)) > h(h^{-1}(x)) = x$ since f and h are increasing, which means that $f(x) < x$; a contradiction. Thus $\text{Fix}(f|_I) \neq \emptyset$ and in particular, we have $\text{Fix}(f) \neq \emptyset$. As $f \neq \text{id}$, so it is not an involution and hence $h \neq \text{id}$.

We will show that $\text{Fix}(h) = \emptyset$. Otherwise, suppose that $\text{Fix}(h) \neq \emptyset$. We will prove that in this case $f = \text{id}$, this leads to a contradiction.

Let $I = (a, b)$ be a connected component of $\mathbb{R} \setminus \text{Fix}(h)$. Then, either a or b is a real number. Let us assume that $a \in \mathbb{R}$. By ([2], Lemma 2.4) f fixes each point of $\text{Fix}(h)$. Then $f(I) = I = h(I)$. Now, by the first step, f has a fixed point $s \in I$. So for each integer $n \in \mathbb{Z}$, $h^n(s) \in I$ and by Lemma 2.2, $f(h^n(s)) = h^n(s)$. Since $\text{Fix}(h|_I) = \emptyset$, we can assume, by swapping h and h^{-1} if necessary, that $h(s) < s < h^{-1}(s)$. Then the points $h^n(s) \in [a, s]$ for $n \in \mathbb{N}$, and accumulate at a . So, f has infinitely many fixed points in the interval $[a, s]$. Since $f \in \text{PL}^+(\mathbb{R})$, there is an integer N such that $f|_{[a, h^N(s)]} = \text{id}$. Thus for any $y \in [a, s]$, one has $f(y) = fh^{-N}(x)$ where $x = h^N(y) \in [a, h^N(s)]$. By Lemma 2.2, $f(y) = h^{-N}f^{(-1)^N}(x) = h^{-N}(x) = y$. Therefore $f|_{[a, s]} = \text{id}$. Similarly, by considering the points $h^{-n}(s) \in [s, b]$, $n \in \mathbb{N}$, we get as above $f|_{[s, b]} = \text{id}$. Therefore $f = \text{id}$ on $\mathbb{R} \setminus \text{Fix}(h)$. As $f = \text{id}$ on $\text{Fix}(h)$, thus $f = \text{id}$ on \mathbb{R} . \square

Proposition 2.4. (1) *If τ is an involution in $\text{PL}^+(\mathbb{S}^1)$ which is not the identity then τ is conjugate in $\text{PL}^+(\mathbb{S}^1)$ to the rotation r_π .*
 (2) *If τ is an involution in $\text{PL}^-(\mathbb{S}^1)$, then it is conjugate in $\text{PL}(\mathbb{S}^1)$ to the reflection s .*

Proof. Let a be the point on \mathbb{S}^1 with coordinates $(1, 0)$ and let b be the point with coordinates $(-1, 0)$. (1) Let $x \in S^1$. Since $\tau \in \text{PL}^+(\mathbb{S}^1)$ and $\tau^2 = \text{id}$, we have $\tau([x, \tau(x)]) = [\tau(x), x]$. Let $v : [a, b] \rightarrow [x, \tau(x)]$ be a piecewise linear homeomorphism, and let ψ be the map of S^1 defined by

$$\psi(x) = \begin{cases} v(x), & \text{if } x \in [a, b]; \\ \tau v r_\pi(x), & \text{if } x \in [b, a]. \end{cases}$$

Then ψ is a well defined piecewise linear homeomorphism of S^1 , and it satisfies $\psi^{-1}\tau\psi = r_\pi$. We conclude that $\tau = \psi r_\pi \psi^{-1}$ is conjugate in $\text{PL}(\mathbb{S}^1)$ to r_π .

(2) Let $\{c, d\} = \text{Fix}(\tau)$. We have $\tau([c, d]) = [d, c]$. Let $u : [a, b] \rightarrow [c, d]$ be a piecewise linear homeomorphism and let φ be the map of \mathbb{S}^1 defined by

$$\varphi(x) = \begin{cases} u(x), & \text{if } x \in [a, b] \\ \tau u s(x), & \text{if } x \in [b, a] \end{cases}$$

Then $\varphi \in \text{PL}(\mathbb{S}^1)$. If $x \in [a, b]$ then $\varphi(s(x)) = \tau u s(s(x)) = \tau u(x) = \tau \varphi(x)$. If $x \in [b, a]$ then $\varphi(s(x)) = u(s(x)) = \tau \varphi(x)$. Therefore $\tau = \varphi s \varphi^{-1}$. \square

Theorem 2.5 ([6]). *Let $h \in \text{PL}^+(\mathbb{S}^1)$ with rotation number $\rho(h)$ irrational. Then h is conjugate in $\text{Homeo}^+(\mathbb{S}^1)$ to the rotation $r_{\rho(h)}$.*

Lemma 2.6. *Let $f \in \text{Homeo}^+(\mathbb{S}^1)$ and $g \in \text{Homeo}(\mathbb{S}^1)$ such that $fg = gf$ and $\text{Fix}(f) \neq \emptyset \neq \text{Fix}(g)$. Then $\text{Fix}(f) \cap \text{Fix}(g) \neq \emptyset$.*

Proof. Let $x \in \text{Fix}(g)$ and let $\omega_f(x)$ be the ω -limit set of x under f . It is well known that $\omega_f(x)$ is a periodic orbit (see [5]). Since $\text{Fix}(f) \neq \emptyset$, $\rho(f) = 0$ and any periodic orbit under f is a fixed point of f . It follows that $\omega_f(x) = \{a\}$; where $a \in \text{Fix}(f)$. As \mathbb{S}^1 is compact, so $(f^n(x))_n$ converges to a . Now, we have $g(f^n(x)) = f^n(g(x)) = f^n(x)$ since g commutes with f . So, $f^n(x)$ converges to $g(a) = a$. Hence $a \in \text{Fix}(f) \cap \text{Fix}(g)$. \square

3. Reversibility in $\text{PL}^+(\mathbb{S}^1)$

3.1. Reversibility in $\text{PL}^+(\mathbb{S}^1)$ of elements $f \in \text{PL}^+(\mathbb{S}^1)$ with $\rho(f) = 0$.
The aim of this subsection is to prove the following proposition.

Proposition 3.1. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. Then f has a lift \tilde{f} in $\text{PL}^+(\mathbb{R})$, which is strongly reversible in $\text{PL}(\mathbb{R})$ by an element of $\text{PL}^-(\mathbb{R})$.*

Proof. Since $\rho(f) = 0$, f has a fixed point $x_0 = e^{it} \in \mathbb{S}^1$. We can assume that $1 \in \text{Fix}(f)$ (by taking $r_{-t}fr_t$ instead of f , where $r_t(z) = e^{it}z$ is the rotation by angle t). Let \tilde{f} be the lift for f such that $\tilde{f}(0) = 0$. Then for all $n \in \mathbb{Z}$, $(T\tilde{f})^n(0) = n$. Let $\alpha_0 : [0, 1] \rightarrow [0, 1]$ be an orientation preserving piecewise linear homeomorphism ($\alpha_0 \in \text{PL}^+([0, 1])$) and for each $n \in \mathbb{Z}$, let $\alpha_n : [n, n+1] \rightarrow [n, n+1]$ be the homeomorphism defined as: $\alpha_n = T^n\alpha_0(T\tilde{f})^{-n}$. Define $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ as $\alpha|_{[n, n+1]} = \alpha_n$, for all $n \in \mathbb{Z}$. Then $\alpha \in \text{PL}^+(\mathbb{R})$ and $T\tilde{f} = \alpha^{-1}T\alpha$. Moreover, $(T\tilde{f})$ is a lift of f . On the other hand, the translation T satisfies $T^{-1} = iTi$; where i is the involution of \mathbb{R} defined by $x \mapsto 1 - x$. Then $(T\tilde{f})^{-1} = \tau(T\tilde{f})\tau$; where $\tau = \alpha^{-1}i\alpha$ is an involution in $\text{PL}^-(\mathbb{R})$. The proof is complete. \square

Proposition 3.2. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is reversed by $h \in \text{PL}^+(\mathbb{S}^1)$, then there exists a lift $\tilde{f} \in \text{PL}^+(\mathbb{R})$ of f which is reversed in $\text{PL}^+(\mathbb{R})$ by any lift \tilde{h} of h .*

Proof. Since $\rho(f) = 0$, $\text{Fix}(f) \neq \emptyset$ and there is a lift $\tilde{f} \in \text{PL}^+(\mathbb{R})$ of f such that $\text{Fix}(\tilde{f}) \neq \emptyset$. We have

$$(3.1) \quad f^{-1} = hfh^{-1}$$

Let $\tilde{h} \in PL^+(\mathbb{R})$ be a lift of h . Then $\tilde{h}\tilde{f}\tilde{h}^{-1}$ is a lift of hfh^{-1} . From equality (3.1), it follows that $\tilde{f}^{-1} = \tilde{h}\tilde{f}\tilde{h}^{-1}T_{-q}$ for some integer $q \in \mathbb{Z}$. As $T\tilde{f} = \tilde{f}T$ and $T\tilde{h} = \tilde{h}T$, then $\tilde{f}^{-1} = T_{-q}\tilde{h}\tilde{f}\tilde{h}^{-1} = \tilde{h}(T_{-q}\tilde{f})\tilde{h}^{-1}$. Therefore, since \tilde{f} has a fixed point, $(T_{-q}\tilde{f})$ has also a fixed point $a \in \mathbb{R}$. Then $(T_{-q}\tilde{f})(a) = a$, equivalently to $\tilde{f}(a) = T_q(a)$. So,

$$\tilde{f}^n(a) = (T_q)^n(a) = a + nq, \quad \forall n \in \mathbb{Z}.$$

Now, we show that $q = 0$. Suppose that $q \neq 0$ (say $q > 0$). Then we have $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [a + nq, a + (n+1)q]$. Let $\alpha_0 : [a, a+q] \rightarrow [0, 1]$ be a piecewise linear orientation preserving homeomorphism such that $\alpha_0(a) = 0$ and $\alpha_0(a+q) = 1$. For $n \in \mathbb{Z}$, let $\alpha_n : [a + nq, a + (n+1)q] \rightarrow [n, n+1]$ be a homeomorphism defined as $\alpha_n = T^n \alpha_0 \tilde{f}^{-n}$. Define $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ as $\alpha|_{[a+nq, a+(n+1)q]} = \alpha_n$ for all $n \in \mathbb{Z}$. Then $\alpha \in PL^+(\mathbb{R})$ and we see that $\tilde{f} = \alpha^{-1}T\alpha$, which is impossible since $\text{Fix}(\tilde{f}) \neq \emptyset$. We conclude that $q = 0$ and so $\tilde{f}^{-1} = \tilde{h}\tilde{f}\tilde{h}^{-1}$. \square

Proposition 3.3. *Let $f \in PL^+(\mathbb{S}^1)$ which is not the identity and such that $\rho(f) = 0$. Then f is reversible in $PL^+(\mathbb{S}^1)$ if and only if some lift \tilde{f} of f is reversed by a homeomorphism $\tilde{h} \in PL^+(\mathbb{R})$ satisfying $\tilde{h}T = T\tilde{h}$.*

Proof. The if part follows from Proposition 3.2. The only if part: let $f \in PL^+(\mathbb{S}^1)$ with $\rho(f) = 0$ and let \tilde{f} be a lift for f such that $\tilde{f}^{-1} = \tilde{h}\tilde{f}\tilde{h}^{-1}$; where $\tilde{h} \in PL^+(\mathbb{R})$ satisfying $\tilde{h}T = T\tilde{h}$. Then \tilde{h} is a lift for the homeomorphism $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ defined by $h(e^{i2\pi x}) = e^{i2\pi\tilde{h}(x)}$, $\forall x \in \mathbb{R}$. Then $h \in PL^+(\mathbb{S}^1)$ and for all $x \in \mathbb{R}$, we have $f^{-1} = hfh^{-1}$. \square

Proposition 3.4. *Let $f \in PL^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is a reversible element in $PL^+(\mathbb{S}^1)$ then some lift \tilde{f} of f is conjugate to a homeomorphism $\tilde{g} \in PL^+(\mathbb{R})$ which is the lift of a homeomorphism $g \in PL^+(\mathbb{S}^1)$ that is reversible by the rotation r_π .*

Proof. From Proposition 3.2, there exists $\tilde{h} \in PL^+(\mathbb{R})$ such that $\tilde{f}^{-1} = \tilde{h}\tilde{f}\tilde{h}^{-1}$. Then by Lemma 2.3, $\text{Fix}(\tilde{h}) = \emptyset$. So \tilde{h} is conjugate in $PL^+(\mathbb{R})$ to either the translation $T : x \mapsto x + 1$ or $T_{-1} : x \mapsto x - 1$, say T for example (see [2]). It follows that \tilde{f} is conjugate to a homeomorphism $\tilde{g} \in PL^+(\mathbb{R})$ satisfying $\tilde{g}^{-1} = T\tilde{g}T^{-1}$. Therefore $T^2\tilde{g} = \tilde{g}T^2$ and we can define a homeomorphism $g \in PL^+(\mathbb{S}^1)$ by $g(e^{i\pi x}) = e^{i\pi\tilde{g}(x)}$, $\forall x \in \mathbb{R}$. We have $g^{-1}(e^{i\pi x}) = e^{i\pi\tilde{g}^{-1}(x)} = e^{i\pi T\tilde{g}T^{-1}(x)} = r_\pi g r_{-\pi}(e^{i\pi x})$, $\forall x \in \mathbb{R}$, which means that $g^{-1} = r_\pi g r_\pi$. \square

Proof of the part (1) of Theorem 1.2. Let f be a reversible homeomorphism in $PL^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is the identity, assertion (1) is clear. So suppose that f is not the identity. Then there exists $h \in PL^+(\mathbb{S}^1)$ such that $f^{-1} = hfh^{-1}$. Let us prove that $\rho(h) \in \mathbb{Q}$. Otherwise, h is conjugate

to an irrational rotation r by Theorem 2.5; there exists $\alpha \in \text{Homeo}^+(\mathbb{S}^1)$ such that $h = \alpha r \alpha^{-1}$. It follows that $f^{-1} = \alpha r \alpha^{-1} f \alpha r^{-1} \alpha^{-1}$, equivalently $g^{-1} = r g r^{-1}$; where $g = \alpha^{-1} f \alpha$. Then by Lemma 2.2 for each integer $n \in \mathbb{Z}$,

$$(3.2) \quad r^{2n} g = g r^{2n}.$$

Since $\rho(f) = 0$, $\text{Fix}(g) \neq \emptyset$. So let $a \in \text{Fix}(g)$. By equality (3.2), for each $n \in \mathbb{Z}$, $r^{2n}(a) \in \text{Fix}(g)$. As the rotation r is irrational, we have $\mathbb{S}^1 = \overline{\{r^{2n}(a) : n \in \mathbb{Z}\}}$. Therefore $\mathbb{S}^1 \subset \text{Fix}(g)$ and $g = \text{id}$. So, $f = \text{id}$, a contradiction. Let then $\rho(h) = \frac{p}{q}$; where p and q are coprime positive

integers. Then h^q has a fixed point. Let us prove that q is even.

Otherwise, by Lemma 2.2, $f^{-1} = h^q f h^{-q}$. Then as in the proof of Proposition 3.2, there is a lift \tilde{h}^q of h^q such that $\text{Fix}(\tilde{h}^q) \neq \emptyset$ and a lift \tilde{f} for f such that $\tilde{f}^{-1} = \tilde{h}^q \tilde{f} \tilde{h}^{-q}$; this contradicts Lemma 2.3. We conclude that q cannot be odd.

Now, one can take $q = 2i$ for some integer i . Since $\rho(h^{2i}) = p$, $\text{Fix}(h^{2i}) \neq \emptyset$. Since $f h^{2i} = h^{2i} f$ and $\text{Fix}(f) \neq \emptyset$, so by Lemma 2.6, there exists $a \in \mathbb{S}^1$ such that $f(a) = a = h^{2i}(a)$. It follows that $f(h^j(a)) = h^j(a)$ for each integer j since $f^{-1} = h f h^{-1}$. Let $\mu : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the homeomorphism defined by

$$\mu(z) = \begin{cases} h^i(z), & \text{if } z \in [a, h^i(a)] \\ h^{-i}(z), & \text{if } z \in [h^i(a), a] \end{cases}$$

It is clear that μ is an involution in $\text{PL}^+(\mathbb{S}^1)$. We have

$$(3.3) \quad \mu f \mu(z) = \begin{cases} h^{-i} f h^i(z), & \text{if } z \in [a, h^i(a)] \\ h^i f h^{-i}(z), & \text{if } z \in [h^i(a), a] \end{cases}$$

We prove that the integer i is odd: Otherwise, suppose that i is even. Then $f h^i = h^i f$, and by (3.3) we obtain that $\mu f \mu = f$. Moreover, as $f^{-1} = h f h^{-1}$, then we have $f^{-1} = h \mu f \mu h^{-1}$. Let $a = e^{i2\pi y}$; where $y \in [0, 1[$, and let \tilde{h} be a lift of h . As $\rho(h) = \frac{p}{2i}$, then $n = 2i$ is the smallest integer such that h^n has a fixed point. Therefore, $\tilde{h}^i(y) \neq y$. Assume that $y < \tilde{h}^i(y)$. Since $\mu(a) = h^i(a)$ and \tilde{h}^i is a lift for h^i , there is a lift $\tilde{\mu}$ for μ such that $\tilde{\mu}(y) = \tilde{h}^i(y)$. By Proposition 3.2, there exists a lift \tilde{f} of f such that $\tilde{f}(y) = y$ and $\tilde{f}^{-1} = \tilde{h} \tilde{\mu} \tilde{f} \tilde{\mu}^{-1} = \tilde{h} \tilde{f} \tilde{h}^{-1}$. It follows that $\tilde{\mu} \tilde{f} \tilde{\mu} = \tilde{f}$. Therefore $\tilde{\mu} \tilde{f} \tilde{\mu}(y) = \tilde{\mu} \tilde{f}(\tilde{h}^i(y)) = \tilde{f}(y) = y$. Since $\tilde{\mu} \tilde{f}$ is an increasing homeomorphism, the inequality $y < \tilde{h}^i(y)$ implies that $\tilde{\mu} \tilde{f}(y) \leq y$; equivalently $\tilde{h}^i(y) \leq y$, which is a contradiction since $y < \tilde{h}^i(y)$. Now the equality $f^{-1} = h f h^{-1}$ implies that $f^{-1} = h^i f h^{-i}$, equivalently $f^{-1} = h^{-i} f h^i$. From (3.3), we deduce that $\mu f \mu = f^{-1}$. Therefore f is strongly reversible by μ in $\text{PL}^+(\mathbb{S}^1)$. This completes our proof. \square

Lemma 3.5. *Let $I = (a, b)$ be an open interval in \mathbb{R} or in \mathbb{S}^1 . Then the following statements hold.*

- (1) *Let $f \in \text{PL}^-(I)$. Then f is reversible in $\text{PL}(I)$ if and only if f is an involution.*
- (2) *Let $f \in \text{PL}^+(I)$. Then f is reversible in $\text{PL}(I)$ by an element of $\text{PL}^-(I)$ if and only if f is strongly reversible in $\text{PL}(I)$ by an element of $\text{PL}^-(I)$.*

Proof. • First, assume that I is an open interval in the real line \mathbb{R} .

Assertion (1). If f is an involution then it is reversible in $\text{PL}(I)$ by the identity map. Conversely, let $f \in \text{PL}^-(I)$, $h \in \text{PL}(I)$ such that $hfh^{-1} = f^{-1}$. By replacing h with hf if necessary, we can assume that $h \in \text{PL}^+(I)$. One can extend f and h on \mathbb{R} as follows:

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in (a, b) \\ a + b - x, & \text{if } x \in \mathbb{R} \setminus (a, b) \end{cases} ; \quad \hat{h}(x) = \begin{cases} h(x), & \text{if } x \in (a, b) \\ x, & \text{if } x \in \mathbb{R} \setminus (a, b) \end{cases}$$

Then, clearly $\hat{f} \in \text{PL}^-(\mathbb{R})$, $\hat{h} \in \text{PL}^+(\mathbb{R})$ and $\hat{h}\hat{f}\hat{h}^{-1} = \hat{f}^{-1}$. So, by ([2], Proposition 2.5), $\hat{f}^2 = \text{id}$ on \mathbb{R} . It follows that $f^2 = \text{id}$ on I .

Assertion (2). Let $f \in \text{PL}^+(I)$ and $h \in \text{PL}^-(I)$ such that $hfh^{-1} = f^{-1}$. Let $\{p\} = \text{Fix}(h)$, $p \in I$. We define the involution $\tau \in \text{PL}^-(I)$ as follows:

$$\tau(x) = \begin{cases} h^{-1}(x), & \text{if } x \geq p \\ h(x), & \text{if } x \leq p \end{cases}$$

If $f(p) = p$, then clearly $\tau f \tau = f^{-1}$. If $f(p) \neq p$, let (c, d) be the connected component of $I \setminus \text{Fix}(f)$ containing p (when $\text{Fix}(f) = \emptyset$, $(c, d) = I$). By the fact that $hfh^{-1} = f^{-1}$, we have $h(\text{Fix}(f)) = \text{Fix}(f)$. Therefore $h((c, d)) = (c, d) = f((c, d))$ (since $h(p) = p$). Let

$$\hat{f}(x) = \begin{cases} f(x), & \text{if } x \in (c, d) \\ x, & \text{if } x \in \mathbb{R} \setminus (c, d) \end{cases}$$

and

$$\hat{h}(x) = \begin{cases} h(x), & \text{if } x \in (c, d) \\ x, & \text{if } x \in \mathbb{R} \setminus (c, d) \end{cases}.$$

Then \hat{h} is one bump function and satisfies $\hat{h}\hat{f}\hat{h}^{-1} = \hat{f}^{-1}$. From ([2], Lemma 4.2), $\hat{h}^2 = \text{id}$ on (c, d) and hence $h(x) = h^{-1}(x)$, for each $x \in (c, d)$. We conclude that the equality $\tau f \tau = f^{-1}$ is satisfied.

• Now, let us assume that $I = (a, b)$ is an open interval in the circle \mathbb{S}^1 . Then, there exists an open interval $\hat{I} = (t_1, t_2)$ in \mathbb{R} such that the map $\varphi : (t_1, t_2) \rightarrow (a, b)$ given by $\varphi(t) = e^{2i\pi t}$, is a homeomorphism. For $f \in \text{PL}(I)$, set $g = \varphi^{-1}f\varphi$. We have $g(t) = \varphi^{-1}f\varphi(t) = \varphi^{-1}f(e^{i2\pi t}) = \varphi^{-1}(e^{i2\pi \tilde{f}(t)}) = \tilde{f}(t)$, for each $t \in (t_1, t_2)$. Hence $g \in \text{PL}(\hat{I})$.

Assertion (1). If $f \in \text{PL}^-(I)$ then $g \in \text{PL}^-(\widehat{I})$. If f is reversed by $h \in \text{PL}(I)$ i.e. $hfh^{-1} = f^{-1}$ then g is reversed by $k = \varphi^{-1}h\varphi \in \text{PL}(\widehat{I})$. Hence by above, g is an involution and so is f .

Assertion (2). If $f \in \text{PL}^+(I)$ and $h \in \text{PL}^-(I)$ such that $f^{-1} = hfh^{-1}$ then g is reversed in $\text{PL}(\widehat{I})$ by an element of $k = \varphi^{-1}h\varphi \in \text{PL}^-(\widehat{I})$. By the above, g is strongly reversible in $\text{PL}(\widehat{I})$ by an element of $\text{PL}^-(\widehat{I})$. Hence there exists an involution $\tau \in \text{PL}^-(\widehat{I})$ such that $\tau g \tau = g^{-1}$. So $(\varphi \tau \varphi^{-1})f(\varphi \tau \varphi^{-1}) = f^{-1}$. As $(\varphi \tau \varphi^{-1}) \in \text{PL}^-(I)$ and is an involution, so f is strongly reversible in $\text{PL}(I)$ by an element of $\text{PL}^-(I)$. \square

Lemma 3.6. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$, then it is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.*

Proof. If f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$, then by Proposition 3.4, there exist $\alpha \in \text{PL}(\mathbb{S}^1)$ and $g \in \text{PL}^+(\mathbb{S}^1)$ such that $f = \alpha g \alpha^{-1}$ and $g^{-1} = r_\pi g r_\pi$. By Proposition 3.2, there exists a lift $\tilde{g} \in \text{PL}^+(\mathbb{R})$ of g such that

$$\tilde{g}^{-1} = T_{\frac{1}{2}} \tilde{g} T_{\frac{1}{2}}; \quad (*)$$

Recall that $T_{\frac{1}{2}}$ defined by $T_{\frac{1}{2}}(t) = t + \frac{1}{2}$ for all $t \in \mathbb{R}$, is a lift of r_π .

Let \tilde{s} be the involution in $\text{PL}^-(\mathbb{R})$ defined by $\tilde{s}(t) = -t$. Then $\tilde{s}T_{\frac{1}{2}}$ is an involution in $\text{PL}^-(\mathbb{R})$, and by equality (*), we have $\tilde{s}\tilde{g}^{-1} = (\tilde{s}T_{\frac{1}{2}})(\tilde{g}\tilde{s})(\tilde{s}T_{\frac{1}{2}})$. Therefore $(\tilde{g}\tilde{s}) \in \text{PL}^-(\mathbb{R})$ which is strongly reversible in $\text{PL}(\mathbb{R})$ by an element of $\text{PL}^-(\mathbb{R})$. So by Lemma 3.5, $(\tilde{g}\tilde{s})^2 = \text{id}$; equivalently $\tilde{g}^{-1} = \tilde{s}\tilde{g}\tilde{s}$. The involution \tilde{s} satisfies $\tilde{s}(t+1) = \tilde{s}(t) - 1$ for all $t \in \mathbb{R}$. Therefore \tilde{s} is a lift for the involution σ of $\text{PL}^-(\mathbb{S}^1)$ defined by

$$\sigma(e^{2i\pi t}) = e^{2i\pi \tilde{s}(t)}; \quad \forall t \in \mathbb{R}.$$

It follows that for each $t \in \mathbb{R}$, $g^{-1}(e^{2i\pi t}) = e^{2i\pi \tilde{s}\tilde{g}\tilde{s}(t)} = \sigma g \sigma(e^{2i\pi t})$. So $g^{-1} = \sigma g \sigma$. We deduce that $f^{-1} = \tau f \tau$; where $\tau = \alpha \sigma \alpha^{-1}$ is an involution in $\text{PL}^-(\mathbb{S}^1)$. \square

3.2. Proof of the part (2) of Theorem 1.2. We need the following lemma.

Lemma 3.7. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\text{Fix}(f^n) \neq \emptyset$ for some integer $n \in \mathbb{N}^*$. Then the following are equivalent.*

- (1) f^n is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.
- (2) f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.

Proof. If f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an involution $\mu \in \text{PL}^-(\mathbb{S}^1)$ then $f^{-1} = \mu f \mu$, which implies that $f^{-n} = \mu f^n \mu$. Conversely, assume that, for some integer $n \in \mathbb{N}^*$, f^n is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an involution $\tau \in \text{PL}^-(\mathbb{S}^1)$, that is $f^{-n} = \tau f^n \tau$. We will show that f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. We have

$$(3.4) \quad (f^{n-1}\tau)f^n(\tau f^{1-n}) = f^{-n},$$

$$(3.5) \quad (f\tau)f^n(\tau f^{-1}) = f^{-n}.$$

The equality (3.4) implies that $(f^{n-1}\tau)^2 f^n = f^n (f^{n-1}\tau)^2$. Therefore from Lemma 2.6, there exists $a \in \text{Fix}(f^n) \cap \text{Fix}((f^{n-1}\tau)^2)$. Thus $f^n(a) = a$, $f^n(\tau(a)) = \tau(a)$ and $f^{n-1}(\tau(a)) = \tau f^{1-n}(a) = \tau f(a)$. In particular, we have $f\tau((\tau(a), f(a))) = (\tau(a), f(a))$. Then the restriction of $f^n/(\tau(a), f(a))$ is an element of $\text{PL}^+((\tau(a), f(a)))$, which is reversed by $(f\tau)|_{(\tau(a), f(a))} \in \text{PL}^-((\tau(a), f(a)))$ by equality (3.5). Then by Lemma 3.5, $f^n|_{(\tau(a), f(a))}$ is strongly reversible in $\text{PL}((\tau(a), f(a)))$ by an involution $\sigma \in \text{PL}^-((\tau(a), f(a)))$; that is, $f^n|_{(\tau(a), f(a))} = \sigma f^n|_{(\tau(a), f(a))} \sigma$. The point $f(a)$ is either in $(a, \tau(a))$ or in $(\tau(a), a)$. We can assume that $f(a) \in (\tau(a), a)$. Since f is orientation-preserving, we can easily see that

$$\mathbb{S}^1 = \bigcup_{p=1}^n [f^p(a), f^p\tau(a)] \cup \bigcup_{p=2}^{n+1} [f^p\tau(a), f^{p+1}(a)].$$

Let $\mu : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the map of \mathbb{S}^1 defined by

$$\mu(x) = \begin{cases} f^{n-p+1}\tau f^{n-p}(x), & \text{if } x \in [f^p(a), f^p\tau(a)], \quad \forall 1 \leq p \leq n; \\ f^{n-p}\sigma f^{n-p}(x), & \text{if } x \in (f^p\tau(a), f^{p+1}(a)), \quad \forall 2 \leq p \leq n+1 \end{cases}$$

The map μ is a well defined homeomorphism of \mathbb{S}^1 . Moreover $\mu \in \text{PL}^-(\mathbb{S}^1)$ that satisfies $\mu^2 = \text{id}$ and $\mu f \mu = f^{-1}$. This completes the proof. \square

Proof of the part (2) of Theorem 1.2 Let $f \in \text{PL}^+(\mathbb{S}^1)$ be reversible in $\text{PL}^+(\mathbb{S}^1)$. Then the rotation number $\rho(f)$ is equal to either 0 or $\frac{1}{2}$. The first case $\rho(f) = 0$ corresponds to the first part of Theorem 1.2. In the second case, $\rho(f) = \frac{1}{2}$ we have $\rho(f^2) = 0 \pmod{1}$. Let us prove that f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. By hypothesis, there exists a homeomorphism $h \in \text{PL}^+(\mathbb{S}^1)$ such that $f^{-1} = h f h^{-1}$ and so $f^{-2} = h f^2 h^{-1}$. Then by the proof of the part (1) of Theorem 1.2, we know that either $\rho(h) \in \mathbb{R} \setminus \mathbb{Q}$ or $\rho(h) = \frac{1}{2i} \pmod{1}$; where i is an odd integer. It follows that in the first case, $f^2 = \text{id}$, and in the second case, f^2 is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$ (see the proof of the part (1) of Theorem 1.2). By Lemma 3.6, f^2 is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. As $\text{Fix} f^2 \neq \emptyset$, we conclude by Lemma 3.7 that f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. \square

4. Reversibility in $\text{PL}(\mathbb{S}^1)$

4.1. Reversibility in $\text{PL}(\mathbb{S}^1)$ of elements of $\text{PL}^+(\mathbb{S}^1)$. The aim of this section is to prove the following proposition.

Proposition 4.1. *Let $f \in \text{PL}^+(\mathbb{S}^1)$. Then f is reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ if and only if it is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.*

Lemma 4.2. (1) *Let $I = (a, b)$ be an open interval in \mathbb{R} or in \mathbb{S}^1 . Then every fixed point free element $v \in \text{PL}^+(I)$ is strongly reversible in $\text{PL}(I)$ by an element of $\text{PL}^-(I)$.*

(2) *Let $f \in \text{PL}^+(\mathbb{S}^1)$. If f has exactly one fixed point, then f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.*

Proof. (1) • Let I be an open interval in \mathbb{R} and $v \in \text{PL}^+(I)$ be a fixed point free homeomorphism. A similar construction as in the proof of ([7], Theorem 1) prove that there exists an involution $\alpha \in \text{PL}^-(I)$ satisfying $v^{-1} = \alpha v \alpha$.

• Assume that $I = (a, b)$ is an open interval in \mathbb{S}^1 . Then, there exists an open interval $\hat{I} = (t_1, t_2)$ in \mathbb{R} such that the map $\varphi : (t_1, t_2) \rightarrow (a, b)$ given by $\varphi(t) = e^{2i\pi t}$, is a homeomorphism. If v is a fixed point free element in $\text{PL}^+(I)$, then $\varphi^{-1}v\varphi$ is a fixed point free element in $\text{PL}^+(\hat{I})$. Then, by the above, there exists an involution $\alpha \in \text{PL}^-(\hat{I})$ satisfying $\varphi^{-1}v^{-1}\varphi = \alpha\varphi^{-1}v\varphi\alpha$. It follows that $v^{-1} = (\varphi\alpha\varphi^{-1})v(\varphi\alpha\varphi^{-1})$. As $\tau = \varphi\alpha\varphi^{-1} \in \text{PL}^-(I)$, we conclude that v is strongly reversible in $\text{PL}(I)$ by an element of $\text{PL}^-(I)$.

(2) Let $\{a\} = \text{Fix}(f)$. By (1), the restriction $f|_{\mathbb{S}^1 \setminus \{a\}}$ is strongly reversible in $\text{PL}(\mathbb{S}^1 \setminus \{a\})$ by an involution $\sigma \in \text{PL}^-(\mathbb{S}^1 \setminus \{a\})$. Then we extend σ to a map $\hat{\sigma} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by

$$\hat{\sigma}(x) = \begin{cases} \sigma(x), & \text{if } x \in \mathbb{S}^1 \setminus \{a\}, \\ a, & \text{if } x = a \end{cases}$$

We see that $\hat{\sigma}$ is an involution in $\text{PL}^-(\mathbb{S}^1)$ which satisfies $f^{-1} = \hat{\sigma}f\hat{\sigma}$. \square

Lemma 4.3. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ then it is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.*

Proof. Assume that there exists $h \in \text{PL}^-(\mathbb{S}^1)$ such that $f^{-1} = hfh^{-1}$. Let us show that f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. If f has exactly one fixed point, then the conclusion follows from Lemma 4.2. Now, assume that f has more than one fixed point. Since $h \in \text{PL}^-(\mathbb{S}^1)$, so h has exactly two fixed points a and b which divides the circle \mathbb{S}^1 onto two

connected components $A = (a, b)$ and $B = (b, a)$ satisfying $h(A) = B$ and $h(B) = A$. Moreover we have always $\text{Fix}(f) \cap A \neq \emptyset$ and $\text{Fix}(f) \cap B \neq \emptyset$. Let c be the nearest point of $\text{Fix}(f) \cap A$ to the point a , and let d be the nearest point of $\text{Fix}(f) \cap A$ to the point b . From the equality $f^{-1} = hfh^{-1}$, we see that $h(c)$ is the nearest point of $\text{Fix}(f) \cap B$ to a and that $h(d)$ is the nearest point of $\text{Fix}(f) \cap B$ to b . The restrictions $f|_{(h(c), c)}$ and $f|_{(d, h(d))}$ are fixed point free piecewise linear homeomorphisms of open arcs of the circle \mathbb{S}^1 . Then by Lemma 4.2, (1), they are reversed respectively in $\text{PL}((h(c), c))$ and $\text{PL}((d, h(d)))$ by involutions $\sigma_1 \in \text{PL}^-(h(c), c)$ and $\sigma_2 \in \text{PL}^-(d, h(d))$. Define $\tau : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ as follows

$$\tau(x) = \begin{cases} h(x), & \text{if } x \in [c, d], \\ h^{-1}(x), & \text{if } x \in [h(d), h(c)] \\ \sigma_1(x), & \text{if } x \in (h(c), c) \\ \sigma_2(x), & \text{if } x \in (d, h(d)) \end{cases}$$

We can easily see that τ is an involution in $\text{PL}^-(\mathbb{S}^1)$ that satisfies $f^{-1} = \tau f \tau$. \square

Proof of Proposition 4.1. Assume that f is reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. We distinguish two cases:

Case 1: $\rho(f) = 0$. Then f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ by Lemma 4.3.

Case 2: $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. Then by Theorem 2.5, there exists $\alpha \in \text{Homeo}(\mathbb{S}^1)$ such that $f = \alpha r \alpha^{-1}$; where r is the rotation of \mathbb{S}^1 by $\rho(f)$. On the other hand, there exists $h \in \text{PL}^-(\mathbb{S}^1)$ such that $f^{-1} = h f h^{-1}$, which implies that $r^{-1} = g r g^{-1}$, where $g = \alpha^{-1} h \alpha$. Then

$$(4.1) \quad g^2 r = r g^2.$$

Since g is an orientation-reversing element of $\text{Homeo}(\mathbb{S}^1)$, $\text{Fix}(g) \neq \emptyset$. Let $a \in \text{Fix}(g) \subset \text{Fix}(g^2)$. The equality (4.1) implies that for each $n \in \mathbb{Z}$, $g^2 r^n = r^n g^2$. It follows that $r^n(a) \in \text{Fix}(g^2)$, for each $n \in \mathbb{Z}$ and by the fact that $\mathbb{S}^1 = \overline{\{r^n(a) : n \in \mathbb{Z}\}}$, we obtain that $\mathbb{S}^1 = \text{Fix}(g^2)$. Thus $g^2 = \text{id}$ and so $h^2 = \text{id}$. We conclude that f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by the involution $h \in \text{PL}^-(\mathbb{S}^1)$.

Case 3. $\rho(f) = \frac{p}{q} \in \mathbb{Q} \setminus \{0\}$. In this case, $\rho(f^q) = 0$ and by the case 1, f^q is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. So by Lemma 3.7, f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. \square

4.2. Reversibility in $\text{PL}(\mathbb{S}^1)$ of elements of $\text{PL}^-(\mathbb{S}^1)$. In this paragraph we study reversibility in $\text{PL}(\mathbb{S}^1)$ of elements of $\text{PL}^-(\mathbb{S}^1)$ by proving the following proposition.

Proposition 4.4. *Let $f \in \text{PL}^-(\mathbb{S}^1)$. Then the following statements are equivalent.*

- (1) f is reversible in $PL(\mathbb{S}^1)$ by an element of $PL^+(\mathbb{S}^1)$.
- (2) f is reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$.
- (3) f is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$.
- (4) f is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^+(\mathbb{S}^1)$.

Lemma 4.5. *Let $f \in PL^-(\mathbb{S}^1)$. Then the following statements are equivalent.*

- (1) f is reversible by an element $h \in PL(\mathbb{S}^1)$ that fixes each of the fixed points of f .
- (2) $f^2 = \text{id}$.

Proof. (1) \implies (2): Since f is orientation-reversing, so it has exactly two fixed points a and b . Set $I = \mathbb{S}^1 \setminus \{a\}$, it is an open interval in \mathbb{S}^1 . As $f^{-1} = hfh^{-1}$ and $h(a) = a$, the restriction $f|_I \in PL^-(I)$ and is reversed by $h|_I \in PL(I)$. Then by Lemma 3.5, (1), $f|_I$ is an involution and so is f .

(2) \implies (1): is clear. \square

Proof of Proposition 4.4. (1) \implies (2): Let $f \in PL^-(\mathbb{S}^1)$, $h \in PL^+(\mathbb{S}^1)$ such that $f^{-1} = hfh^{-1}$. So $f^{-1} = (fh)f(h^{-1}f^{-1})$. Hence f is reversed in $PL(\mathbb{S}^1)$ by $fh \in PL^-(\mathbb{S}^1)$.

(2) \implies (3): Let $f \in PL^-(\mathbb{S}^1)$ and $h \in PL^-(\mathbb{S}^1)$ be such that

$$f^{-1} = hfh^{-1}. \quad (*)$$

Since f is orientation-reversing, so it has exactly two fixed points a and b . We have $h(\text{Fix}(f)) = \text{Fix}(f)$. So, either h fixes each of a and b or it interchanges them. In the first case, we have $f^2 = \text{id}$ by Lemma 4.5. So f is an involution in $PL^-(\mathbb{S}^1)$ and hence it is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$. In the second case; that is $h(a) = b$ and $h(b) = a$, we have $h((a, b)) = (a, b)$. So by equality (*), the restriction $f|_{(a, b)}^2$ is an element of $PL^+((a, b))$ that is reversed by $h|_{(a, b)}$. Thus, by Lemma 3.5, (2), $f|_{(a, b)}^2$ is strongly reversible in $PL((a, b))$ by an involution $\tau \in PL^-((a, b))$; that is,

$$f|_{(a, b)}^{-2} = \tau f|_{(a, b)}^2 \tau. \quad (**)$$

Let $\mu : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the map defined by

$$\mu(x) = \begin{cases} \tau(x), & \text{if } x \in [a, b] \\ f^{-1}\tau f^{-1}(x), & \text{if } x \in [b, a] \end{cases}$$

Clearly $\mu \in PL^-(\mathbb{S}^1)$ and $\mu f \mu = f^{-1}$. Moreover, by equality (**), we have $\mu^2 = \text{id}$. This implies that f is strongly reversible in $PL(\mathbb{S}^1)$ by an element of $PL^-(\mathbb{S}^1)$.

(3) \implies (4). Assume that $f^{-1} = \tau f \tau$, where τ is an involution in $PL^-(\mathbb{S}^1)$. Then $(f\tau)^2 = \text{id}$ and so $f^{-1} = (f\tau)f(\tau f^{-1})$. Hence f is also strongly reversible in $PL(\mathbb{S}^1)$ by the involution $(f\tau)$ in $PL^+(\mathbb{S}^1)$.

(4) \implies (1) is clear. \square

Proof of Theorem 1.4. This follows from Theorem 1.2, Propositions 4.1 and 4.4. \square

Remark 4.6. In Proposition 4.4, we showed that any reversible element in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ must be strongly reversible. This does not hold for elements of $\text{Homeo}^-(\mathbb{S}^1)$ (see [1]).

5. Strong reversibility in $\text{PL}(\mathbb{S}^1)$ of elements of $\text{PL}^+(\mathbb{S}^1)$

5.1. Strong reversibility of elements of $\text{PL}^+(\mathbb{S}^1)$. The aim of this subsection is to prove the part (1) of Theorem 1.5.

Lemma 5.1. *Let $f, g \in \text{PL}^+(\mathbb{S}^1)$ such that $\text{Fix}(f) \neq \emptyset \neq \text{Fix}(g)$. If $\Delta_f = \Delta_g$, then there exists $v \in \text{PL}^+(\mathbb{S}^1)$ such that $g = vfv^{-1}$ and $v = \text{id}$ on $\text{Fix}(f)$.*

Proof. Since $\Delta_f = \Delta_g$, we have $\text{Fix}(f) = \text{Fix}(g)$. For each open interval component (a, b) of $\mathbb{S}^1 \setminus \text{Fix}(f)$, there exists an orientation preserving piecewise linear homeomorphism $u : (a, b) \rightarrow \mathbb{R}$. Then ufu^{-1} and ugu^{-1} are two fixed point free elements of $\text{PL}^+(\mathbb{R})$. Since $\Delta_f = \Delta_g$, ufu^{-1} and ugu^{-1} are conjugate in $\text{PL}^+(\mathbb{R})$ by ([2], Proposition 2.6). Let $v_0 \in \text{PL}^+((a, b))$ such that $g(x) = v_0 f v_0^{-1}(x)$ for $x \in (a, b)$. Then the map v defined by $v(x) = v_0(x)$ for $x \in (a, b)$ and $v(x) = x$ for $x \in \text{Fix}(f)$, is the required homeomorphism. \square

Lemma 5.2. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = \frac{1}{2}$. Then f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$ if and only if $f^2 = \text{id}$.*

Proof. Lemma 5.2 is a particular case of ([1], Theorem 3.3). \square

Proof of the part (1) of Theorem 1.5. Assume that f is a strongly reversible element of $\text{PL}^+(\mathbb{S}^1)$. We know that either $\rho(f) = 0$ or $\rho(f) = \frac{1}{2}$. If $\rho(f) = \frac{1}{2}$, then by Lemma 5.1, $f^2 = \text{id}$. If $\rho(f) = 0$, then $\text{Fix}(f) \neq \emptyset$, and since f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$, there exists an involution $h \in \text{PL}^+(\mathbb{S}^1)$ such that $f^{-1} = h^{-1}fh$. Therefore, $\rho(h) = \frac{1}{2}$ and by ([1], Lemma 2.1), $\Delta_f = -\Delta_f \circ h$.

Conversely, assume that $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\text{Fix}(f) \neq \emptyset$ and there exists $h \in \text{PL}^+(\mathbb{S}^1)$ with $\rho(h) = \frac{1}{2}$ satisfying $\Delta_f = -\Delta_f \circ h$. Then $\Delta_{f^{-1}} = \Delta_{h^{-1}fh}$. By Lemma 5.1, there exists $v \in \text{PL}^+(\mathbb{S}^1)$ such that $f^{-1} = v^{-1}h^{-1}fhv$; which means that f is reversible by $hv \in \text{PL}^+(\mathbb{S}^1)$. Since $\text{Fix}(f) \neq \emptyset$, $\rho(f) = 0$ and by Theorem 1.2, f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$. If $f^2 = \text{id}$, then it is clear that f is strongly reversible in $\text{PL}^+(\mathbb{S}^1)$ by the identity map. \square

5.2. Strong reversibility of elements of $\text{PL}^-(\mathbb{S}^1)$. The aim of this subsection is to prove the part (2) of Theorem 1.5.

Lemma 5.3. *Let $f \in \text{PL}^+(\mathbb{S}^1)$ with $\rho(f) = 0$. Then f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ if and only if there exists $h \in \text{PL}^-(\mathbb{S}^1)$ such that $\Delta_f = \Delta_f \circ h$.*

Proof. Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) = 0$. If f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ then there exists an involution $h \in \text{PL}^-(\mathbb{S}^1)$ such that $f^{-1} = h^{-1}fh$. Thus by Lemma 2.1, $\Delta_f = -\deg(h)\Delta_f \circ h = \Delta_f \circ h$. Conversely, if $\Delta_f = \Delta_f \circ h$ for some element $h \in \text{PL}^-(\mathbb{S}^1)$ then $\Delta_f = -\deg(h)\Delta_f \circ h$ and $\Delta_{f^{-1}} = \Delta_{h^{-1}fh}$. So, by the proof of the part (1) of Theorem 1.5, f is reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. By Proposition 4.1, f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. \square

Proof of the part (2) of Theorem 1.5. (i): Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) \in \mathbb{Q}$. If f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$, then there exists an involution $h \in \text{PL}^-(\mathbb{S}^1)$ such that $f^{-1} = h^{-1}fh$, which implies that $f^{-n_f} = h^{-1}f^{n_f}h$. So by Lemma 5.3, we have $\Delta_{f^{n_f}} = \Delta_{f^{n_f}} \circ h$. Conversely, assume that there exists $h \in \text{PL}^-(\mathbb{S}^1)$ such that $\Delta_{f^{n_f}} = \Delta_{f^{n_f}} \circ h$. Then by Lemma 5.3, f^{n_f} is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ and by Lemma 3.7, f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$.

(ii): Let $f \in \text{PL}^+(\mathbb{S}^1)$ such that $\rho(f) \in \mathbb{R} \setminus \mathbb{Q}$. If f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$ then there exists an involution $\tau \in \text{PL}^-(\mathbb{S}^1)$ such that $f^{-1} = \tau f \tau$. On the other hand, by Theorem 2.5, there is $h \in \text{Homeo}^+(\mathbb{S}^1)$ such that $f = hr_{\rho(f)}h^{-1}$. Therefore $hr_{\rho(f)}^{-1}h^{-1} = \tau hr_{\rho(f)}h^{-1}\tau$. As $r_{\rho(f)}^{-1} = sr_{\rho(f)}s$, where $s : z \mapsto \bar{z}$ is the reflection, so $h^{-1}\tau hsr_{\rho(f)} = r_{\rho(f)}h^{-1}\tau hs$. Hence $h^{-1}\tau hs = r_t$ for some $t \in \mathbb{R}$. It follows that $\tau = hr_tsh^{-1} \in \text{PL}^-(\mathbb{S}^1)$.

Conversely, if there is $h \in \text{Homeo}^+(\mathbb{S}^1)$ such that $f = hr_{\rho(f)}h^{-1}$ where h satisfies $hrsh^{-1} \in \text{PL}^-(\mathbb{S}^1)$, for some rotation r of \mathbb{S}^1 , then $\tau = hrsh^{-1}$ is an involution in $\text{PL}^-(\mathbb{S}^1)$ and satisfies $f^{-1} = \tau f \tau$ (since $rs = sr^{-1}$). Therefore f is strongly reversible in $\text{PL}(\mathbb{S}^1)$ by an element of $\text{PL}^-(\mathbb{S}^1)$. \square

6. Proof of Theorem 1.6

Proof of (i). If $f^2 = \text{id}$, there is nothing to prove. If $f^2 \neq \text{id}$, from Theorem 1.5.(1), it suffices to find two involutions τ and h in $\text{PL}^+(\mathbb{S}^1)$ such that $\text{Fix}(\tau f) \neq \emptyset$ and $\Delta_{\tau f} = -\Delta_{\tau f} \circ h$ since in that case, f is a composition of three involutions of $\text{PL}^+(\mathbb{S}^1)$. There is a point x in \mathbb{S}^1 such that $x \neq f^2(x)$. We can assume that the points x , $f(x)$ and $f^2(x)$ occur in that order anticlockwise around \mathbb{S}^1 . Choose a point y in $(x, f(x))$ such that $f^{-1}(y)$

be in $(f^2(x), x)$. Let $u : [x, f(x)] \longrightarrow [f(x), x]$ be an orientation-preserving piecewise linear homeomorphism such that:

$$u(y) = f(y),$$

$$\begin{aligned} u(t) &< \min(f(t), f^{-1}(t)), & \text{for } t \in (x, y); \\ f(t) &< u(t) < f^{-1}(t), & \text{for } t \in (y, f(x)). \end{aligned}$$

Then, let τ be the involution in $\text{PL}^+(\mathbb{S}^1)$ defined by

$$\tau(t) = \begin{cases} u(t), & \text{if } t \in [x, f(x)] \\ u^{-1}(t), & \text{if } t \in [f(x), x] \end{cases}$$

We have $\tau f(t) = t$ if and only if $t = x$ or $t = y$. So, $\text{Fix}(\tau f) = \{x, y\}$. Moreover, we have:

$$\begin{aligned} \forall t \in (x, y), \quad f(t) > u(t) &\iff \tau f(t) > t \\ \forall t \in (y, f(x)), \quad f(t) < u(t) &\iff \tau f(t) < t \\ \forall t \in (f(x), x), \quad u(u^{-1}(t)) < f^{-1}(u^{-1}(t)) &\iff \tau f(t) < t. \end{aligned}$$

Therefore:

$$\Delta_{\tau f}(t) = \begin{cases} 0, & \text{if } t = x, y \\ 1, & \text{if } t \in (x, y) \\ -1, & \text{if } t \in (y, x). \end{cases}$$

Now, let $v : [x, y] \longrightarrow [y, x]$ be any orientation-preserving piecewise linear homeomorphism, and let h be the involution in $\text{PL}^+(\mathbb{S}^1)$ defined by

$$h(t) = \begin{cases} v(t), & \text{if } t \in [x, y] \\ v^{-1}(t), & \text{if } t \in [y, x] \end{cases}$$

It is easy to see that h satisfies $\Delta_{\tau f} \circ h = -\Delta_{\tau f}$. We conclude that each member of $\text{PL}^+(\mathbb{S}^1)$ can be expressed as a composite of three involutions of $\text{PL}^+(\mathbb{S}^1)$. So, $\text{PL}^+(\mathbb{S}^1) = I_3(\text{PL}^+(\mathbb{S}^1)) = R_2(\text{PL}^+(\mathbb{S}^1))$. There are elements in $\text{PL}^+(\mathbb{S}^1)$ which are not strongly reversible in $\text{PL}^+(\mathbb{S}^1)$; one can choose, for example, a homeomorphism $f \in \text{PL}^+(\mathbb{S}^1)$ which is not an involution and with rotation number $\rho(f) = \frac{1}{2}$ (such map f is not strongly reversible in $\text{PL}^+(\mathbb{S}^1)$ by Lemma 5.1). The fact that $R_1(\text{PL}^+(\mathbb{S}^1)) \subsetneq I_2(\text{PL}^+(\mathbb{S}^1))$ follows from Theorem 1.2.

Proof of (ii). If $f \in \text{PL}^+(\mathbb{S}^1)$, then by (i), $f \in I_3(\text{PL}^+(\mathbb{S}^1))$. If $f \in \text{PL}^-(\mathbb{S}^1)$, then $\text{Fix}(f) \neq \emptyset$. Let $a \in \text{Fix}(f)$ and let $I = \mathbb{S}^1 \setminus \{a\}$. Then, there exists an open interval \hat{I} in \mathbb{R} such that the map $\varphi : \hat{I} \longrightarrow I$ given by $\varphi(t) = e^{2i\pi t}$, is a homeomorphism. Set $g = \varphi^{-1}f\varphi$. We have $g \in \text{PL}^-(\hat{I})$. Choose an involution $\sigma \in \text{PL}^-(\hat{I})$ such that, for each $x \in \hat{I}$, $\sigma(x) > g(x)$. Then $g(\sigma(x)) < x$, for each $x \in \hat{I}$. Therefore, $g\sigma$ is a fixed point free element in $\text{PL}^+(\hat{I})$. By Lemma 4.2, (1), it is strongly reversible in $\text{PL}(\hat{I})$ by an element of $\text{PL}^-(\hat{I})$; which means that there exist three involutions u, v in $\text{PL}^-(\hat{I})$ such that $g\sigma = uv$. Thus $g = uv\sigma$. It follows that $f|_I = \varphi g \varphi^{-1} = (\varphi u \varphi^{-1})(\varphi v \varphi^{-1})(\varphi \sigma \varphi^{-1})$. By extending $\varphi u \varphi^{-1}$, $\varphi v \varphi^{-1}$ and $\varphi \sigma \varphi^{-1}$ to \mathbb{S}^1 by fixing a , we get three involutions τ_1, τ_2 and τ_3 in $\text{PL}^-(\mathbb{S}^1)$ satisfying $f = \tau_1 \tau_2 \tau_3$. Hence $f \in I_3(\text{PL}(\mathbb{S}^1))$. We conclude that

$\text{PL}(\mathbb{S}^1) = I_3(\text{PL}(\mathbb{S}^1)) = R_2(\text{PL}(\mathbb{S}^1))$. Moreover, $\text{PL}(\mathbb{S}^1) \neq I_2(\text{PL}(\mathbb{S}^1))$ as in the proof of (i). From Theorem 1.4, we have $R_1(\text{PL}(\mathbb{S}^1)) = I_2(\text{PL}(\mathbb{S}^1))$. Now to show that $I_2(\text{PL}(\mathbb{S}^1)) \neq I_1(\text{PL}(\mathbb{S}^1))$, it suffices to choose a nontrivial reversible element f in $\text{PL}^+(\mathbb{S}^1)$ which is not the identity and with rotation number $\rho(f) = 0$. \square

Remark 6.1. Contrarily to $\text{PL}^-(\mathbb{R})$ (cf. Lemma 3.5), there exists an element of $\text{PL}^-(\mathbb{S}^1)$ which is strongly reversible in $\text{PL}(\mathbb{S}^1)$ but not an involution.

Proof. Indeed, Suppose that the remark is not true. We will prove in this case that any element $f \in \text{PL}^-(\mathbb{S}^1)$ is an involution, this leads to a contradiction since there are elements in $\text{PL}^-(\mathbb{S}^1)$ which are not involutions. Indeed, let σ be any involution in $\text{PL}^-(\mathbb{S}^1)$. Then $\sigma f \in \text{PL}^+(\mathbb{S}^1)$ and from Theorem 1.6, (i), there exist three involutions τ_1 , τ_2 and τ_3 in $\text{PL}^+(\mathbb{S}^1)$ such that $\sigma f = \tau_1 \tau_2 \tau_3$. This implies that $f = \sigma \tau_1 \tau_2 \tau_3$. By assumption, $\sigma \tau_1$ is an involution in $\text{PL}^-(\mathbb{S}^1)$ and then so is $(\sigma \tau_1) \tau_2$. We conclude that f is an involution. \square

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